

Quantum Error-Correcting Codes with Preexisting Protected Qubits

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We provide a systematic way of constructing entanglement-assisted quantum error-correcting codes via graph states in the scenario of preexisting perfectly protected qubits. It turns out that the preexisting entanglement can help beat the quantum Hamming bound and can enhance (not only behave as an assistance) the performance of the quantum error correction. Furthermore we generalize the error models to the case of not-so-perfectly-protected qubits and introduce the quantity *infidelity* as a figure of merit and show that our code outperforms also the ordinary quantum error-correcting codes.

The quantum error-correcting code (QECC) [1, 2, 3, 4, 5] is an active way to deal with the errors caused by the quantum noises during the process of quantum communication and quantum computation. Simply speaking, a QECC is just a subspace that corrects errors. The first quantum code is the well known Shor's 9-qubit code [1], which is a quantum analog of the classical repetition code followed by Steane's 7-qubit code [2] and the optimal 5-qubit code [3, 4]. Along with the establishment of stabilizer formalism [6, 7, 8] in QECC theory, various more efficient quantum codes were constructed.

The constructions of QECCs depend on the error models. Standard QECCs deal with the error model in which every qubit may equally go wrong. Often there may exist some special qubits that are protected from noises somehow, e.g., in a communication scenario in which Alice want to send some qubits via a noisy quantum channel to Bob while they may share some ideal EPR pairs beforehand. In that case the qubits in Bob's hand are free from errors caused by the noisy channel. For another instance, in the nuclear spin-electron spin system, the error probability of nuclear spin is as 10^{-6} as that of electron spin. Or some of our physical qubits are protected by some QECCs already. It is therefore reasonable to assume that some physical qubits in which our quantum data is encoded are perfectly protected from errors. The entanglement-assisted QECC (EAQECC) [9] deals with exactly such a situation where the perfectly protected qubits are ensured by the preexisting EPR pairs. It is a special example of QECCs with preexisting protected qubits dealing with an error model in which there are some physical qubits that suffer errors with a smaller probability than other physical qubits.

Recently a graphical approach to the construction of QECCs [10] has been developed in the cases of both stabilizer and nonadditive codes [11, 12], binary and nonbinary codes [13, 14]. For the binary case a codeword stabilized codes approach [15] has also been introduced. In this Letter we shall at first present a graphical construction of all the EAQECCs and then discuss in details two special codes found via our approach: a family of codes beating the quantum Hamming bound and a 9-qubit code that demonstrates the fact that the entanglement can really enhance the performance of the QECCs and not only behave as an assistance. At last we con-

sider the error model where the qubits are not so perfectly protected and introduce the concept of *infidelity* to characterize the performance of a code in such an error model and discuss the advantage of our 9-qubit code beyond EAQECC.

Graphical constructions of EAQECCs Instead of in a communication scenario, we shall develop our graphical approach to the EAQECCs in the scenario where, among $n + e$ physical qubits, there exist e pure qubits that suffer no error at all during the whole quantum process. As we will note later there is a slightly difference (one-way classical communications) between these two scenarios.

Considering a graph $G = (V, \Gamma)$ composed of a vertex set V with $n + e$ vertices and edges specified by an *adjacency matrix* Γ , which is a symmetric matrix with vanishing diagonal entries and $\Gamma_{ab} = 1$ if a, b are connected and $\Gamma_{ab} = 0$ otherwise. Let $N_a = \{b \in V | \Gamma_{ab} = 1\}$ denote the neighborhood of vertex a and P be a subset of V containing e vertices. We label a system of $n + e$ qubits with V and the pure qubits with P . The graph state [16, 17, 18] on G reads

$$|\Gamma\rangle = \prod_{a,b \in V} (\mathcal{U}_{ab})^{\Gamma_{ab}} |+\rangle_x^V, \quad (1)$$

where \mathcal{U}_{ab} is the controlled phase gate between qubit a and b , i.e., $\mathcal{U}_{ab} = (1 + \mathcal{Z}_a + \mathcal{Z}_b - \mathcal{Z}_a \mathcal{Z}_b)/2$ and $|+\rangle_x^V$ is the joint $+1$ eigenstate of all \mathcal{X}_a ($a \in V$) with $\mathcal{Z}, \mathcal{X}, \mathcal{Y}$ denoting three Pauli matrices. The graph state $|\Gamma\rangle$ is also the joint $+1$ eigenstate of $n + e$ vertex stabilizers

$$\mathcal{G}_a = \mathcal{X}_a \prod_{b \in N_a} \mathcal{Z}_b, \quad (a \in V) \equiv \mathcal{X}_a \mathcal{Z}_{N_a}. \quad (2)$$

Obviously $\mathcal{G}_S = \prod_{a \in S} \mathcal{G}_a$ stabilizes also the graph state for arbitrary $S \subseteq V$. By specifying a collection of K different vertex subsets $\{C_i\}_{i=1}^K$ the graph state basis $\{|\Gamma_{C_i}\rangle \equiv \mathcal{Z}_{C_i} |\Gamma\rangle\}_{i=1}^K$ spans a K dimensional subspace, where we have denoted $\mathcal{Z}_C = \prod_{a \in C} \mathcal{Z}_a$.

Given a graph G on the $n + e$ vertices and an integer $1 \leq d \leq n$ we define a (d, e) -purity set as

$$\mathbb{S}_d = \{S \subseteq V | (S \cup N_S) \cap P = \emptyset, |S \cup N_S| < d\} \quad (3)$$

and a (d, e) -uncoverable set as

$$\mathbb{D}_d = 2^V - \{\delta \triangle N_\omega | (\delta \cup \omega) \cap P = \emptyset, |\delta \cup \omega| < d\}. \quad (4)$$

Here we have denoted by $A \Delta B = A \cup B - A \cap B$ the symmetric difference of two subsets A and B , by $|S|$ the number of the elements in set S , and by $N_S = \Delta_{v \in S} N_v$ the neighborhood of a vertex subset S .

A coding clique \mathbb{C}_d^K of a given graph G with a pure points set P is a collection of K vertex subsets that satisfies:

- i) $\emptyset \in \mathbb{C}_d^K$
- ii) $|S \cap C|$ is even for all $S \in \mathbb{S}_d$ and $C \in \mathbb{C}_d^K$
- iii) $C \Delta C' \in \mathbb{D}_d$ for all $C, C' \in \mathbb{C}_d^K$

We denote by $(G, K, d; e)$ as the subspace spanned by graph state basis $\{|\Gamma_C\rangle | C \in \mathbb{C}_d^K\}$. If the coding cliques form a group with respect to the symmetric difference, then we call the coding clique as coding group and denote the corresponding subspace as $[G, k, d; e]$ with $K = 2^k$. As in Ref.[9] we denote by $[[n, k, d; e]]$ an EAQECC of length n and distance d with pre-existing e pure qubits. As usual we also denote by $[[n, k, d]]$ a standard stabilizer code on n qubits of distance d . We have

Theorem *The subspace $(G, K, d; e)$ is an EAQECC $((n, K, d; e))$ and $[G, k, d; e]$ is an $[[n, k, d; e]]$ code.*

Proof. It is enough to prove that for any error \mathcal{E}_d that acts nontrivially on less than d impure qubits we can get $\langle \Gamma_C | \mathcal{E}_d | \Gamma_{C'} \rangle = f(\mathcal{E}_d) \delta_{CC'}$ [5][3] for all $C, C' \in \mathbb{C}_d^K$. Without lose of generosity we assume that $\mathcal{E}_d = \mathcal{X}_\omega \mathcal{Z}_\delta$ for some pair of subsets δ, ω with $(\delta \cup \omega) \cap P = \emptyset$ and $|\delta \cup \omega| < d$, which represents that there are \mathcal{X} , \mathcal{Y} , and \mathcal{Z} errors on the qubits in $\omega - \delta \cap \omega$, $\omega \cap \delta$, and $\delta - \omega \cap \delta$ respectively. When acting on the graph state the error $\mathcal{E}_d \propto \mathcal{G}_\omega \mathcal{Z}_\Omega$ can be replaced by phase flip errors \mathcal{Z}_Ω on $\Omega := \delta \Delta N_\omega$. If Ω is empty then $\delta = N_\omega$ and the error is proportional to \mathcal{G}_ω . In this case we have $\mathcal{G}_\omega | \Gamma_C \rangle = | \Gamma_C \rangle$ for all $C \in \mathbb{C}_d^K$ because $|\omega \cap C|$ is even which stems from the fact that $|\omega \cup N_\omega| < d$, i.e., $\omega \in \mathbb{S}_p$ and Condition 1. Thus the error behaves like a constant operator on the coding subspace and can be neglected. If Ω is not empty then $\Omega \notin \mathbb{D}_d \cup \emptyset$ because it is covered by (δ, ω) and $|\delta \cup \omega| < d$. As a result $\langle \Gamma_C | \mathcal{E}_d | \Gamma_{C'} \rangle \propto \langle \Gamma | \mathcal{Z}_{C \Delta C'} \mathcal{Z}_\Omega | \Gamma \rangle = 0$ for all $C, C' \in \mathbb{C}_d^K$ because condition 2 ensures that $C \Delta C' \neq \Omega$. Now we have proved the first part of the theorem. Further more, if we have a k dimensional coding group of a graph G which is generated by $\langle C_1, C_2, \dots, C_k \rangle$, then we have a code $(G, 2^k, d)$ according to the proof above. Since k constraints $|S \cup C_i| = \text{even}$ for $i = 1, 2, \dots, k$ have exactly $n - k$ independent solutions $\langle S_1, S_2, \dots, S_{n-k} \rangle$, the stabilizer of the code is generated by $\langle \mathcal{G}_{S_i} \rangle_{i=1}^{n-k}$. Q.E.D.

Since the EAQECCs are a stabilizer code on $n + e$ qubits and all the stabilizer code can be constructed in the graphical way [10], it follows that all the EAQECCs can also be found in this graphical way.

According to this theorem, we can construct EAQECC systematically as follows. First, input a graph $G = (V, \Gamma)$ on $n + e$ vertices. Second, choose a distance d and compute the (d, e) -purity set \mathbb{S}_d and (d, e) -uncoverable set \mathbb{D}_d . Third, find all the K-clique \mathbb{C}_d^K [19], and then for every clique we obtain a $(G, K, d; e)$ code, i.e., an $((n, K, d; e))$ code. And if the coding

clique form a group with respect to the symmetric difference, we will have an stabilizer $[[n, k, d; e]]$ code.

In Table I we have listed the best EAQECCs we have found, giving the distance d as a function of the block size $n + e$ and number of encoded qubits with $e = 1$. The entries with an asterisk mark the improvements over the best former EAQECC.

$(n+e) \setminus k$	1	2	3	4	5	6	7	8	9	10
3	2	1	1							
4	3*	2	1	1						
5	4*	2	2	1	1					
6	5*	3*	2	2	1	1				
7	6*	3*	2	2	2	1	1			
8	7*	3	3	2	2	2	1	1		
9	8*	3	3	3*	2	2	2	1	1	
10	9*	4	3	3	3*	2	2	2	1	1

TABLE I: $e=1$

EAQECCs beating the quantum Hamming bound The simplest EAQECC found via our graphical approach is the 1-error correcting code $[[3, 1, 3; 1]]$ which encodes 1 logical qubit by 3+1 physical qubits, including one pure qubit. In comparison, to encode one logical qubit at least 4+1 or 3+2 or 5 physical qubits have to be used in the known EAQECC and the standard code $[[5, 1, 3]]$.

We consider the star graph S_4 on 4 vertices as shown in Fig.1(a) and denote the corresponding graph state on 4 qubits as $|S_4\rangle$. Here we have supposed that the qubit labeled with 0 suffers no error at all. Given $d = 3$ and $e = 1$ we obtain that (d, e) -purity set is empty and the graph S_4 admits a coding group with 2 elements $\{\emptyset, \{1, 2, 3\}\}$. The subspace spanned by

$$\{|S_4\rangle, \mathcal{Z}_1 \mathcal{Z}_2 \mathcal{Z}_3 |S_4\rangle\} \quad (5)$$

is the $[[3, 1, 3; 1]]$ code. Three generators of the stabilizer of the code and the syndromes for all 9 single qubit errors are listed in Table II. We see that the code is not pure since all

0	1	2	3		$\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$	\mathcal{Z}_1	\mathcal{Z}_2	\mathcal{Z}_3	\mathcal{Y}_1	\mathcal{Y}_2	\mathcal{Y}_3
\mathcal{X}	\mathcal{Z}	\mathcal{Z}	\mathcal{Z}	\mathcal{G}_0	-	+	+	+	-	-	-
\mathcal{I}	\mathcal{X}	\mathcal{X}	\mathcal{I}	$\mathcal{G}_1 \mathcal{G}_2$	+	-	-	+	-	-	+
\mathcal{I}	\mathcal{X}	\mathcal{I}	\mathcal{X}	$\mathcal{G}_1 \mathcal{G}_3$	+	-	+	-	-	+	-

TABLE II: The stabilizer of the code $[[3, 1, 3; 1]]$.

three bit flip errors \mathcal{X}_k ($k = 1, 2, 3$) give rise to the same syndrome. As a result the quantum Hamming bound [20], which imposes $2^k(3n+1) \leq 2^{n+e}$ on 1-error-correcting EAQECCs of length $n + e$, is violated by our $[[3, 1, 3; 1]]$ code where $n = 3$ and $k = e = 1$.

In fact for any integer n we are able to construct the code $[[n, 1, n; 1]]$ with the star graph S_{n+1} on $n + 1$ qubits labeled

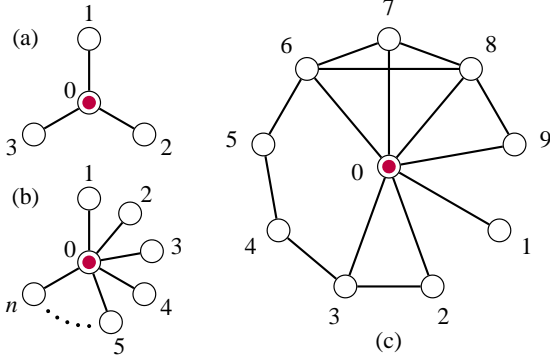


FIG. 1: (a) The star graph for the code $[[3, 1, 3; 1]]$. (b) The star graph on $n + 1$ vertices. (c) The coffeepot-like graph for the code $[[9, 5, 3; 1]]$. Here all the purple vertices label the pure qubits.

with numbers from 0 to n with the central qubit 0 labeling the pure qubit. This is the graphical code $[S_{n+1}, 1, n; 1]$ with coding group generated by $V - \{0\}$ and stabilizer generated by \mathcal{G}_0 and $\mathcal{G}_1\mathcal{G}_j$ with $j = 2, 3, \dots, n$. It can be checked that all this family of code except $n = 4$ violate the quantum Hamming bound $\sum_{s=0}^t 3^s C_n^s \leq 2^{n+e-k}$ with $t = \lfloor \frac{d-1}{2} \rfloor$ for the EAQEC $[[n, k, d; e]]$ since $3^t C_n^t > 2^n$ for $n = d > 4$. It should be noticed that one-way classical communications are needed to use the code $[[n, 1, n; 1]]$ in the communication scenario. However with the presence of a one-way classical communication channel the code $[[n, 1, n; 1]]$ is trivial because Alice and Bob can transfer a qubit safely by teleportation [21].

An entanglement enhanced QECC We consider the coffeepot-like graph T on 10 vertices as shown in Fig.1(c) and denote by $|T\rangle$ the corresponding 10-qubit graph state. We suppose that qubit 0 is perfectly protected from errors and label it with a purple vertex. For $d = 3$ and $e = 1$ the graph T admits a 5-dimensional coding group generated by

$$\{\{1, 5, 7\}, \{2, 4, 7\}, \{3, 4, 9\}, \{3, 6, 7\}, \{4, 5, 8\}\}. \quad (6)$$

Accordingly the 2^5 -dimensional subspace spanned by the graph-state basis $(\mu_1, \mu_2, \dots, \mu_5 = 0, 1)$

$$\mathcal{Z}_1^{\mu_1} \mathcal{Z}_2^{\mu_2} \mathcal{Z}_3^{\mu_3+\mu_4} \mathcal{Z}_4^{\mu_2+\mu_3+\mu_5} \mathcal{Z}_5^{\mu_1+\mu_5} \mathcal{Z}_6^{\mu_4} \mathcal{Z}_7^{\mu_2+\mu_4} \mathcal{Z}_8^{\mu_5} \mathcal{Z}_9^{\mu_3} |T\rangle \quad (7)$$

is a $[[9, 5, 3; 1]]$ code whose stabilizer is generated by five graph stabilizers as listed in Table III. It is obvious that all

n	0	1	2	3	4	5	6	7	8	9
\mathcal{G}_0	X	Z	Z	Z	I	I	Z	Z	Z	Z
$\mathcal{G}_1\mathcal{G}_5\mathcal{G}_8$	I	X	I	I	Z	X	I	Z	X	Z
$\mathcal{G}_2\mathcal{G}_4\mathcal{G}_8\mathcal{G}_9$	Z	I	X	I	X	Z	Z	Z	\mathcal{Y}	\mathcal{Y}
$\mathcal{G}_3\mathcal{G}_6\mathcal{G}_9$	Z	I	Z	X	Z	Z	X	Z	I	X
$\mathcal{G}_1\mathcal{G}_2\mathcal{G}_6\mathcal{G}_7$	I	X	X	Z	I	Z	\mathcal{Y}	\mathcal{Y}	I	I

TABLE III: The stabilizer of the code $[[9, 5, 3; 1]]$.

single-qubit errors will lead to different syndromes except three pairs of errors $\{Z_0, X_1\}$, $\{X_0, Z_9\}$, and $\{Y_0, Y_6\}$. Therefore these could not be corrected if qubit 0 were not perfectly protected from errors.

All the EAQECs known so far either are identical to some QECCs, *e.g.*, the code $[[7, 3, 3; 1]]$ can be constructed from the stabilizer code $[[8, 3, 3]]$, or are equivalent to protocols including standard QECCs plus teleportation, *e.g.*, a $[[5, 2, 3; 1]]$ code can be constructed via the standard code $[[5, 1, 3]]$ without using the protected qubit together with encoding a logical qubit with the pure qubit, or in a communication scenario, teleporting one qubit with the preexisting ideal EPR pair. The $[[9, 5, 3; 1]]$ code constructed above will outperform both the standard QECC and the QECC+teleportation, which can encode at most 4 logical qubits.

We consider at first the communication scenario: Alice and Bob share beforehand an ideal EPR pair and there is a noisy quantum channel between them and Alice can send 9 qubits down the channel and it is assumed that only 1 qubit in the channel will suffer errors which is arbitrary and unknown.

By using an optimal stabilizer code $[[10, 4, 3]]$ [8], Alice can encode 4 logical qubit in 10 qubits and send 9 qubits, keeping her qubit in the ideal EPR pair, down the noisy channel to Bob. After receive 9 qubits from Alice Bob can decode 4 logical qubits by measuring a set of generators of the stabilize as in Table III on 10 qubits in his hand. As an alternative, Alice can also use an optimal 9-qubit stabilizer code $[[9, 3, 3]]$ to encode 3 logical qubits send those 9 qubits down the noisy channel to Bob and then teleport one qubit to Bob (one-way classical communications are needed). In both protocols at most 4 qubits can be encoded.

On the other hand if Alice use the code $[[9, 5, 3; 1]]$ instead she can send 5 logical qubits to Bob. At first it obvious that Alice and Bob can build the graph state $|T\rangle$ by local operations with preexisting one ideal EPR pair. By local operations Alice can also encode 5 logical qubits in 10 qubits in her hand and then send 9 qubits to Bob. It should be noticed that Alice can encode the logical qubits without using her qubit in the EPR pair. In this way Bob decode 5 qubits for 9 qubits Alice sent him and one qubit in EPR pair. We see that in this EAQEC, the ideal EPR pair does not only achieve its own task — ensuring 1 qubit free from errors, but also enhances the encoding ability of the other 9 qubits.

And then we consider the scenario of entanglement purification [22] with one-way classical communications: Alice and Bob share 9 copies of EPR among which 1 copy may go wrong but they do not know which one and one ideal EPR pair. The best protocol without the preexisting ideal EPR pair is to use the $[[9, 3, 3]]$ code (may also use $((9, 12, 3))$ [12]) from which 3 ideal EPR pairs can be purified. However if Alice and Bob measured the stabilizer of the $[[9, 5, 3; 1]]$ code instead on all 10 EPR pairs, they can obtain 5 ideal EPR pairs. After extracting the preexisting ideal EPR pair, they still have 4 ideal EPR pairs left. It is equivalent to use a powerful $[[9, 4, 3]]$ code which does not exist.

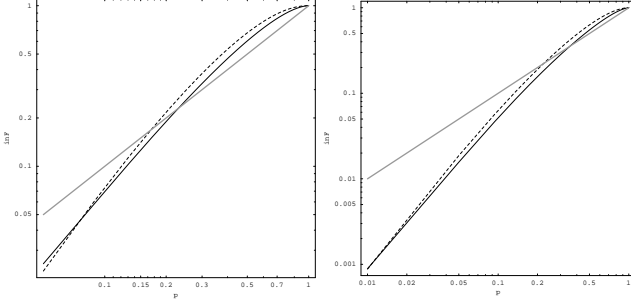


FIG. 2: The infidelities of two codes $[[10, 4, 3]]$ represented by the dashed curve and $[[9, 5, 3; 1]]$ represented by the solid curve in the case of $p_e = p$ on lefthand side and $p_e = p/10$ on the righthand side. The solid gray line represents the infidelity in the case of no code is used.

Good-qubit-assisted QECCs In the discussions above, we have assumed an ideal error model in which there preexist some perfectly protected qubits. It is however more realistic to consider the error model in which there are some physical qubits with a smaller error probability p_e than the error probability p of other qubits. In this situation, we shall demonstrate that the good-qubit-assisted QECC (GQAQECC) will outperform the standard optimal codes as well. To do so we shall introduce a reasonable figure of merit, namely the *infidelity*, to evaluate the performance of a code in addition to the distance d within the error model described above.

A QECC works perfectly only when correctable errors occur and we denote by P_C the probability of effective coding, i.e., the probability of the occurrence of correctable errors. To compare codes encoding different number of logical qubits, we image that we used k qubits directly instead of some code as logical qubits with some error probability p' of physical qubits. This no-coding scheme works perfect only if there is no error at all, i.e., the probability of effective coding is $(1 - p')^k$. To achieve the same probability of effective coding as that of a QECC encoding k logical qubits, the error probability p' of the physical qubits must be

$$\text{inF} = 1 - (P_C)^{\frac{1}{k}} \quad (8)$$

which is defined here to be the infidelity of a QECC. The smaller the infidelity the better the code will perform. For examples infidelities for the EAQECC $[[9, 5, 3; 1]]$ and an optimal stabilizer code $[[10, 4, 3]]$ read respectively

$$\text{inF}_{[[9,5,3;1]]} = 1 - (\bar{p}_e \bar{p}^9 + 9p\bar{p}_e \bar{p}^8)^{\frac{1}{5}}, \quad (9)$$

$$\text{inF}_{[[10,4,3]]} = 1 - (\bar{p}^9 + 9p\bar{p}_e \bar{p}^8)^{\frac{1}{4}}, \quad (10)$$

where $\bar{p} = 1 - p$ and $\bar{p}_e = 1 - p_e$. In Fig.2 we have plotted these two infidelities as functions of p in the case of $p_e = p$ and $p_e = p/10$. We see that even in the symmetric case there are some regions of p that the entanglement enhanced code performs better than the best standard QECC by encoding 1 more logical qubit and with a less infidelity.

Conclusion With preexisting perfectly protected qubits we have demonstrated that there are more efficient QECCs by constructing explicitly a family of 1-error correcting codes violating the quantum Hamming bound and a 9-qubit entanglement enhanced QECC that outperforms the optimal standard QECC in both the communication scenario and entanglement purification scenario. Within a more realistic error model where there are qubits with smaller error probability than other physical qubits, the GQAQECC performs also better than the standard QECC basing one the infidelity as the figure of merit.

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